Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 6, pp. 791–799, 2005

INVARIANT AND PARTIALLY INVARIANT SOLUTIONS OF THE GREEN–NAGHDI EQUATIONS

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UDC 517.9; 532.592

All invariant and partially invariant solutions of the Green–Naghdi equations are obtained that describe the second approximation of shallow water theory. It is proved that all nontrivial invariant solutions belong to one of the following types: Galilean-invariant, stationary, and self-similar solutions. The Galilean-invariant solutions are described by the solutions of the second Painleve equation, the stationary solutions by elliptic functions, and the self-similar solutions by the solutions of the system of ordinary differential equations of the fourth order. It is shown that all partially invariant solutions reduce to invariant solutions.

Key words: *Green–Naghdi equations, invariant and partially invariant solutions, Painleve equation.*

1. Main Results. One of the conventional models of the second-order shallow water approximation is the Green–Naghdi system of equations [1, 2]:

$$h_t + (uh)_x = 0, \qquad u_t + uu_x + gh_x = (h^3(u_{xt} + uu_{xx} - u_x^2))_x/(3h). \tag{1.1}$$

In (1.1), $h(t,x)$ and $u(t,x)$ are used to denote the height of the fluid free surface above the horizontal bottom and the average flow velocity in the horizontal direction; $g = \text{const}$ is the acceleration of gravity.

The basis of the Lie algebra admitted by Eqs. (1.1) is formed by the operators

$$Y_1 = \partial_t, \qquad Y_2 = \partial_x, \qquad Y_3 = t\partial_x + \partial_u, \qquad Y_4 = t\,\partial_t + 2x\,\partial_x + u\,\partial_u + 2h\,\partial_h. \tag{1.2}$$

In the present paper, it is shown that all invariant solutions of Eqs. (1.1) belong to one of the following types.

1. The Galilean-invariant solutions:

— Solutions generated by the subalgebra $\langle Y_1 + \beta Y_3\rangle$ with a real-valued parameter $\beta \neq 0$, which are represented as

$$u = \beta t + U(y), \qquad h = H(y), \qquad UH = c_1, \tag{1.3}$$

where $y = x - \beta t^2/2$ and $c_1 = \text{const}$. Let

$$w = (\beta c_1/\sqrt{3})^{1/3}/U, \qquad y = (c_1^2/(3\beta))^{1/3}\xi - c_2,$$

$\alpha = -\sqrt{3}g/(4\beta)$ and c_2 are constants. Then, the factor-equation describing these solution is written as

$$w\,\frac{d^2w}{d\xi^2} = \frac{1}{2}\Big(\frac{dw}{d\xi}\Big)^2 - \frac{1}{2} - \xi w^2 + 4\alpha w^3; \tag{1.4}$$

— Solutions generated by the subalgebra $\langle Y_3\rangle$ and defined by the formulas

$$u = (x + u_0)/t, \qquad h = h_0/t, \tag{1.5}$$

where u_0 and h_0 are arbitrary constants;

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— Self-similar solutions generated by the subalgebra $\langle Y_1 + \beta Y_3, Y_4\rangle$ and defined by the formulas

$$u = \beta t, \qquad h = (\beta^2/(2g))(t^2 - 2x/\beta), \quad \beta \neq 0. \tag{1.6}$$

2. The stationary solutions generated by the subalgebra $\langle Y_1\rangle$ and represented as

$$u = U(x), \qquad h = H(x), \qquad UH = c_1. \tag{1.7}$$

The factor-equation describing such solutions reduces to the elliptical integral

$$\int_{h_0}^{H} (-3gc_1^{-2}H^3 + c_2H^2 + c_3H + 3)^{-1/2}\,dH = c_4 \pm x, \tag{1.8}$$

where c_1, c_2, c_3, and c_4 are constants. From (1.8) setting $c_1 = j\sqrt{g}$, $c_2 = 3 + 6/j^2$, $c_3 = -6 - 3/j^2$, $c_4 = 0$, $h_0 = j^2$, and $|j| > 1$, we obtain the one-soliton solution (which was found in [1]) of the Green–Naghdi equations:

$$u = j\sqrt{g}(j^2 + (1 - j^2)\tanh^2\xi)^{-1}, \qquad h = j^2 + (1 - j^2)\tanh^2\xi, \qquad \xi = (\sqrt{3(j^2 - 1)}/(2j))x.$$

3. The self-similar solutions generated by the subalgebra $\langle Y_4\rangle$ and represented as

$$u = tU(z), \qquad h = t^2H(z), \qquad z = xt^{-2}. \tag{1.9}$$

The factor-system for the functions U and H has the form

$$\begin{gathered}(UH)' - 2zH' + 2H = 0,\\ 3H(g + (U - 2z)U' + U) + (H^3(U' + U'^2 + (2z - U)U''))' = 0.\end{gathered} \tag{1.10}$$

This system has the particular one-parameter solution $U = c_1$, $H = (c_1/g)(c_1/2 - z)$. In addition to solutions (1.3)–(1.10) there are trivial solutions, for which $u = u_0 = \text{const}$ and $h = h_0 = \text{const}$; in this case, it is possible that $u_0 = 0$ or $h_0 = 0$.

All partially invariant solutions of Eqs. (1.1) reduce to invariant solutions, and, therefore, they are trivial or belong to one of the above-mentioned types.

Equation (1.4) is included in the Painleve list of 50 equations, whose solution dose not have movable singular points different from poles [3]. Its solution w is expressed in terms of the solution v of the second Painleve equation

$$\frac{d^2v}{d\xi^2} = 2v^3 + \xi v - 2\alpha - \frac{1}{2}$$

using the Miura transformation

$$2\alpha w = \frac{dv}{d\xi} + v^2 + \frac{\xi}{2}.$$

The system of Green–Naghdi equations (1.1) admits the Lie algebra (1.2), which is isomorphic to the symmetry algebra of the Korteweg–de Vries equation. The invariant solutions (1.3)–(1.10) have the corresponding analogs for this equation. The question of the existence of multisoliton solutions for the Green–Naghdi equations (1.1) remains open today.

In the subsequent sections of the paper, it is proved that solutions (1.3)–(1.10) exhaust the set of invariant and partially invariant solutions of Eqs. (1.1). The possible representations of the invariant and partially invariant solutions are constructed using the optimal system of subalgebras of the symmetry algebra L_4 (1.2), and the factors-systems obtained in this case are then analyzed.

2. Optimal System of Subalgebras of the Symmetry Algebra [4, 5]. The optimal system (OS) of subalgebras of the algebra L_4 was constructed in [6] along with the OS for all types of three and four-dimensional Lie algebras, and it has the following form:

— the OS of the three-dimensional subalgebras $\Theta_3 L_4$

$$\langle 1, 2, 3\rangle, \quad \langle 2, \alpha 1 + \beta 3, 4\rangle; \tag{2.1}$$

— the OS of the two-dimensional subalgebras $\Theta_2 L_4$

$$\langle 2, \alpha 1 + \beta 3\rangle, \quad \langle 2, 4\rangle, \quad \langle \alpha 1 + \beta 3, 4\rangle; \tag{2.2}$$

TABLE 1

Subalgebra number	Basis of subalgebra	Normalizer of the subalgebra
4.1	1, 2, 3, 4	= 4.1
3.1	1, 2, 3	4.1
3.2	$1+\beta 3$, 2, 4	= 3.2
3.3	2, 3, 4	= 3.3
2.1	2, 3	4.1
2.2	2, $1+\beta 3$	4.1
2.3	2, 4	= 2.3
2.4	$1+\beta 3$, 4	= 2.4
2.5	3, 4	= 2.5
1.1	2	4.1
1.2	$1+\beta 3$	3.2
1.3	3	3.3
1.4	4	= 1.4

Note. The "=" sign denotes self-normalized subalgebras.

— the OS of the one-dimensional subalgebras $\Theta_1 L_4$

$$\langle 2\rangle, \quad \langle \alpha 1+\beta 3\rangle, \quad \langle 4\rangle. \tag{2.3}$$

Here $\alpha, \beta \in \mathbb{R}$ and $\alpha^2+\beta^2 \neq 0$. In formulas (2.1)–(2.3), operators (1.2) are denoted by their numbers for brevity. In calculating the invariants of the subalgebras containing the operator $\alpha 1+\beta 3$, it is more convenient to consider the cases $\alpha = 0$ and $\alpha \neq 0$ separately. Therefore, the OS of subalgebras of the algebra L_4 (1.2) used in this paper differs from (2.1)–(2.3) and is given in Table 1. In this algebra, the subalgebras are enumerated by pairs of numbers $(r.i)$, where r denotes the dimension of the subalgebra, and i its ordinal number among the subalgebras of the given dimension. Below, the invariant solution constructed by the subalgebra $L_{r.i}$ is denoted by IS ($L_{r.i}$), and the partially invariant solution by PIS ($L_{r.i}$).

3. Invariant Solutions. The invariant solutions (ISs) can be treated as partially invariant solutions (PISs) whose defect is equal to zero. The type of PIS is determined by the pair of numbers (ρ, δ), where ρ is the rank of the solution (the number of independent variables on which the invariant functions depend) and δ is the defect of the solution (the number of superfluous functions which do not have invariant representations). Let n and m be the numbers of independent and dependent variables, respectively, of the basic equation. The subalgebra $L_{r.i}$ generates PISs of the type of (ρ, δ), where ρ and δ satisfy the conditions [4]

$$\max\{r_* - n, m-q, 0\} \leq \delta \leq \min\{r_*, m-1\}, \qquad \rho = l - m + \delta. \tag{3.1}$$

Here r_* is the overall rank of the matrix composed of the coordinates of the operators that form the basis of the subalgebra $L_{r.i}$, $l = n+m-r_*$ is the number of functionally independent invariants, and q is the degree of completeness of this set of invariants with respect to the dependent variables.

Table 2 gives the complete set of functionally independent invariants for the subalgebras from the optimal system of subalgebras of the algebra L_4. An analysis of condition (3.1) for them leads to the conclusion that the subalgebras $L_{3.i}$ generate PISs of type (0,1), the subalgebra $L_{2.1}$ generates PISs of type (1,1), subalgebra $L_{2.i}$ ($i = 2, \ldots, 5$) PISs of type (1,1) and ISs of rank 0, and the subalgebras $L_{1.i}$ generate ISs of rank 1. The third column of Table 2 gives the representation of the invariant solution. Here and below, u_0, h_0, a, $c_j = \text{const}$; $y = x - \beta t^2/2$, and $z = x/t^2$.

Let us write all invariant solutions of the Green–Naghdi equations (1.1).

IS ($L_{2.2}$). The factor-system contains an identity and the equality $\beta = 0$, and the solution $u = u_0$, $h = h_0$ exists only for $\beta = 0$.

IS ($L_{2.3}$). The reduced equations (1.1) are written as $2h_0 t = 0$, $u_0 = 0$, and the solution has the form $u = 0$, $h = 0$.

IS ($L_{2.4}$). The factor-system consists of the equations $3u_0 h_0 y^{1/2}/2 = 0$ and $\beta + u_0^2/2 + gh_0 + u_0^2 h_0^2/3 = 0$, whose solution is $u_0 = 0$, $h_0 = -\beta/g$; the invariant solution has the form (1.6). In addition, for $\beta < 0$, there is the solution $u = \beta t + \sqrt{-2\beta}\,(x - \beta t^2/2)^{1/2}$, $h = 0$.

TABLE 2

Subalgebra number	Basis of invariants	Representation of IS	Representation of PIS
3.1	h	—	$h = h_0$, $u = u(t, x)$
3.2	$(u - \beta t)h^{-1/2}$	—	$h = h(t, x)$, $u = \beta t + u_0 h^{1/2}$
3.3	$t^{-2}h$	—	$h = h_0 t^2$, $u = u(t, x)$
2.1	t, h	—	$h = H(t)$, $u = u(t, x)$
2.2	$u - \beta t$, h	$u = \beta t + u_0$, $h = h_0$	$h = h(t, x)$, $u = \beta t + U(h)$
2.3	$t^{-1}u$, $t^{-2}h$	$u = u_0 t$, $h = h_0 t^2$	$h = t^2 H(t, x)$, $u = tU(H)$
2.4	$y^{-1/2}(u - \beta t)$, $y^{-1}h$	$u = \beta t + u_0 y^{1/2}$, $h = h_0 y$	$h = yH(t, x)$, $u = \beta t + y^{1/2}U(H)$
2.5	$t^{-1}u - t^{-2}x$, $t^{-2}h$	$u = u_0 t + x/t$, $h = h_0 t^2$	$h = t^2 H(t, x)$, $u = tU(H) + x/t$
1.1	t, u, h	$u = U(t)$, $h = H(t)$	—
1.2	y, $u - \beta t$, h	$u = \beta t + U(y)$, $h = H(y)$	—
1.3	t, $u - x/t$, h	$u = U(t) + x/t$, $h = H(t)$	—
1.4	z, $t^{-1}u$, $t^{-2}h$	$u = tU(z)$, $h = t^2 H(z)$	—

Note. The dash denotes the absence of solutions of the given type.

IS ($L_{2.5}$). The reduced equations (1.1) are written as $3h_0 t = 0$ and $2u_0 = 0$; the solution has the form $u = x/t$, $h = 0$.

IS ($L_{1.1}$). Substitution of the representation of the solution into Eqs. (1.1) leads to $H' = 0$ and $U' = 0$, which gives the constant invariant solution $u = u_0$, $h = h_0$.

IS ($L_{1.2}$). Substitution of the representation of the solution into Eqs. (1.1) leads to the factor-system

$$(HU)' = 0, \qquad \beta + UU' + gH' = (H^3(UU'' - U'^2))'/(3H),$$

which is twice integrated:

$$U = c_1/H, \qquad HH'' = H'^2/2 - 3/2 - 3(gH^3 + \beta(y + c_2)H^2)/c_1^2. \tag{3.2}$$

For $\beta \neq 0$, the transformation $y = (c_1^2/3\beta)^{1/3}\xi - c_2$, $H = (\sqrt{3}c_1^2/\beta)^{1/3}w$ reduces Eq. (3.2) in the function $H(y)$ to Eq. (1.4) in the function $w(\xi)$.

In the case $\beta = 0$, we obtain the stationary solution (1.7) of Eqs. (1.1).

IS ($L_{1.3}$). The reduced equations (1.1) $H' + H/t = 0$ and $U' + U/t = 0$ are integrated as $U = u_0/t$ and $H = h_0/t$. We obtain the invariant solution (1.5).

IS ($L_{1.4}$). Substitution of the representation of the solution into Eqs. (1.1) leads to the factor-system (1.10).

Let us write the factor-system of the given invariant solution in other equivalent forms because they will be needed to study the partially invariant solutions.

The hodograph transformation $U = U(H)$, $z = v(H)$ reduces (1.10) to the system of equations

$$2Hv' - 2v + HU' + U = 0,$$

$$Uv' + (U - 2v)U' + g + \frac{1}{3H}\Big(H^3\Big(\frac{U'}{v'} + \frac{U'^2}{v'^2} + (2v - U)\Big(\frac{U''}{v'^2} - \frac{U'v''}{v'^3}\Big)\Big)\Big)' = 0$$

in the functions $U(H)$ and $v(H)$. From the first equation, we express $v(H) = -\dfrac{U}{2} - H\displaystyle\int \frac{U}{H^2}\,dH$, and the second equation then becomes

$$U - 2HU' + \frac{1}{\Delta}(g - HU'^2) + \frac{1}{\Delta^2}\Big(\frac{2}{3}H^3U''' + 3H^2U'' + HU'\Big)$$

$$+ \frac{1}{\Delta^3}\Big(\frac{2}{3}H^3U'U''' + H^3U''^2 + \frac{11}{2}H^2U'U'' + \frac{10}{3}HU'^2\Big)$$

$$+ \frac{U'}{\Delta^4}\Big(\frac{H^3}{6}U'U''' + H^3U''^2 + 4H^2U'U'' + 3HU'^2\Big) + \frac{HU'^2}{\Delta^5}\Big(\frac{H}{2}U'' + U'\Big)^2 = 0, \tag{3.3}$$

where $\Delta = v'(H) = -\dfrac{U'}{2} - \dfrac{U}{H} - \displaystyle\int \frac{U}{H^2}\,dH$.

Substitution of the representation of the solution in the form $u=\beta t+(x-\beta t^2/2)^{1/2}U(\bar z)$, $h=(x-\beta t^2/2)H(\bar z)$, $\bar z = t(x-\beta t^2/2)^{-1/2}$ into Eqs. (1.1) leads to the equations

$$3HU + 2H' - \bar z(HU)' = 0,$$

$$\beta + \frac{U^2}{2} + \Big(1 - \frac{\bar z}{2}\,U\Big)U' + g\Big(h - \frac{\bar z}{2}\,H'\Big)$$

$$= \frac{\bar z^5}{12H}\Big(H^3\Big(\frac{U''}{\bar z^3} + \frac{U^2}{\bar z^4} - \frac{3U}{2\bar z^3}U' + \frac{1}{2\bar z^2}\,(U'^2 - UU'')\Big)\Big)'.$$

The transformation $U = U(H)$, $\bar z = v(H)$ reduces them to the system

$$3HUv' - (HU' + U)v + 2 = 0,$$

$$\Big(\beta + \frac{U^2}{2}\Big)v' + \Big(1 - \frac{vU}{2}\Big)U' + g\Big(hv' - \frac{v}{2}\Big)$$

$$+ \frac{v^5}{12H}\Big(H^3\Big(\frac{U'v''}{v^3v'^3} - \frac{U''}{v^3v'^2} - \frac{U^2}{v^4} + \frac{3}{2}\,\frac{UU'}{v^3v'} - \frac{UU'v''}{2v^2v'^3} + \frac{1}{2v^2v'^2}\,(UU'' - U'^2)\Big)\Big)' = 0.$$

From the first equation, we express $v(H) = -\dfrac{2}{3}\,(HU)^{1/3}\displaystyle\int (HU)^{-4/3}\,dH$, substitute it into the second equation, and obtain the following equation for the function $U(H)$:

$$\beta - U^2 + \frac{H^2U^2}{3} + gH - \frac{3g}{2\Delta}\,HUv + \frac{3}{\Delta}\,HUU' + \frac{3U^2}{2\Delta}\,(Uv-2) +$$

$$+\frac{H^2U^2}{8\Delta}\,(18 - 23Uv) + \frac{H^2U^2}{4\Delta^2}\,(94 - 149Uv + 52U^2v^2)$$

$$+ \frac{H^2U^2}{\Delta^3}\Big(80 - \frac{389}{2}\,Uv + 135U^2v^2 - \frac{327}{8}\,U^3v^3\Big) + \frac{21}{2}\,\frac{H^2}{\Delta^4}\,U^2(Uv-2)^3(3Uv-1)$$

$$+ \frac{2}{\Delta^3}\,H^3UU'(\Delta - 1) + \frac{9}{8}\,\frac{H^5}{\Delta^4}\,U^3v^2U'''(Uv-2)^2 - \frac{27}{8}\,\frac{H^6}{\Delta^5}\,U^3v^3U''^2(Uv-2)^2$$

$$+ \frac{H^4}{\Delta^3}\,U^2vU''\Big((Uv-2)\Big(\frac{39}{8}\,Uv - 3\Big) - \frac{1}{\Delta}\,(Uv-2)^2\Big(\frac{99}{8}\,Uv - \frac{21}{4}\Big) + \frac{27}{2\Delta^2}\,Uv(Uv-2)^3\Big)$$

$$- \frac{27}{2}\,\frac{H^2}{\Delta^5}\,U^3v(Uv-2)^4 = 0, \qquad \Delta = (HU)'v - 2. \tag{3.4}$$

Finally, substituting the solution of the form $u = x/t + tU(z)$, $h = t^2H(z)$ into Eqs. (1.1), we obtain the reduced equations

$$3H - zH' + (HU)' = 0,$$

$$2U + (U - z)U' + gH' + (H^3(2 + 3U' + U'^2 + (z-U)U''))'/(3H) = 0.$$

Under the transformation $U = U(H)$, $z = v(H)$, they become

$$3Hv' - v + HU' + U = 0,$$

$$2Uv' + (U - v)U' + g + \frac{1}{3H}\Big(H^3\Big(2 + 3\,\frac{U'}{v'} + \frac{U'^2}{v'^2} + (v-U)\Big(\frac{U''}{v'^2} - \frac{U'v''}{v'^3}\Big)\Big)\Big)' = 0.$$

From the first equation, we express $v(H) = -\dfrac{U}{3} - \dfrac{4}{9}\,H^{1/3}\displaystyle\int H^{-4/3}U\,dH$, substitute it into the second equation, and obtain the following equation for the function $U(H)$:

$$2U-3HU'+\frac{1}{\Delta}\,(g+2H-HU'^2)+\frac{1}{\Delta^2}\Big(H^3U'''+\frac{19}{3}\,H^2U''+\frac{55}{9}\,HU'\Big)$$
$$+\frac{1}{\Delta^3}\Big(\frac{2}{3}\,H^3U'U'''+H^3U''^2+\frac{65}{9}\,H^2U'U''+\frac{173}{27}\,HU'^2\Big)$$
$$+\frac{U'}{\Delta^4}\Big(\frac{H^3}{9}\,U'U'''+\frac{2}{3}\,H^3U''^2+\frac{82}{27}\,H^2U'U''+\frac{74}{27}\,HU'^2\Big)+\frac{HU'^2}{9\Delta^5}\,(HU''+2U')^2=0, \tag{3.5}$$
where $\Delta=v'(H)=-\dfrac{U'}{3}-\dfrac{4}{9}\dfrac{U}{H}-\dfrac{4}{27}\,H^{-2/3}\displaystyle\int H^{-4/3}U\,dH.$

4. Partially Invariant Solutions. The fourth column of Table 2 gives the representation of the partially invariant solution for the two-dimensional and three-dimensional subalgebras of the Lie algebra L_4 (1.2). We note that the PISs constructed by the subalgebras $L_{2.i}$ ($i=2,\dots,5$) are irregular. All PISs of the Green–Naghdi equations are reducible to the invariant solutions of these equations.

In studies of the partially invariant solutions of the Green–Naghdi equations in some cases, if u is chosen as a superfluous function, the factor-system has a particular solution that corresponds to the solution of Eqs. (1.1):
$$h=0,\qquad x=tu+\Phi(u).$$
Below, in the integrating of the factor-system, this single nonreducible solution will be discarded since it has no physical meaning.

PIS ($L_{3.1}$). For $h_0\neq 0$, the factor-system
$$h_0u_x=0,\quad u_t+uu_x=h_0^2(u_{xt}+uu_{xx}-u_x^2)_x/3$$
implies that $u_t=0$ and $u_x=0$. We obtain the constant solution of Eqs. (1.1) $u=u_0$, $h=h_0$, i.e., IS ($L_{2.2}$) with $\beta=0$.

PIS ($L_{3.2}$). Substitution of the representation of the solution into Eqs. (1.1) leads to the system
$$h_t+(\beta t+3u_0h^{1/2}/2)h_x=0,$$
$$\beta+(g+u_0^2/2)h_x+u_0h^{-1/2}(h_t+\beta th_x)/2$$
$$=u_0(h^{3/2}(hh_{xt}-h_th_x/2)+\beta th^{3/2}(hh_{xx}-h_x^2/2)+u_0h^2(hh_{xx}-h_x^2))_x/(6h).$$
For $h\neq 0$, replacing in the second equation the derivatives h_t and h_{xt} by virtue of the first equation, we obtain
$$\beta+(g-u_0^2/4)h_x+u_0^2(h^3h_{xx}+2h^2h_x^2)_x/(12h)=0.$$
In the following, the second equation of the factor-system will be given in this form containing only the derivatives of the required function with respect to x.

The first equation of the factor-system is integrated. Substitution of its solution $x-\beta t^2/2-3u_0th^{1/2}/2=\Phi(h)$ into the second equation leads to the fourth-order polynomial in t:
$$\beta(\Phi'+3u_0th^{-1/2}/4)^5+(g-u_0^2/4)(\Phi'+3u_0th^{-1/2}/4)^4+u_0^2P_2(t)/12=0. \tag{4.1}$$
Here and below, $P_n(t)$ denotes a certain polynomial in t of order n. In Eq. (4.1), splitting in the powers of the variable t is possible. For $\beta\neq 0$, equating the coefficients at the powers of t, we obtain the equations $u_0=0$ and $\Phi'^4(\beta\Phi'+g)=0$. The condition $\Phi'=0$, i.e., $\Phi(h)=\text{const}$, implies the contradictory equality $x-\beta t^2/2=\text{const}$. Therefore, $\Phi(h)=-gh/\beta-a$, which gives the solution of Eqs. (1.1) $u=\beta t$, $h=(\beta/g)(\beta t^2/2-x-a)$, which is invariant with respect to a subalgebra with the operator basis $\langle Y_1+\beta Y_3, Y_4+2aY_2\rangle$. This subalgebra is similar to the subalgebra $L_{2.2}$.

For $\beta=0$, equating the coefficients at the powers of t in (4.1), we obtain $u_0=0$ and $\Phi'=0$, which implies the contradictory equality $x=\text{const}$. However, in the case $\beta=0$, the factor-system has a constant solution. Thus, we obtain the solution $u=u_0h_0^{1/2}$, $h=h_0$, which is reducible to the IS ($L_{2.2}$) with $\beta=0$.

PIS ($L_{3.3}$). For $h_0\neq 0$, the first equation of the factor-system
$$h_0(2+tu_x)=0,\qquad u_t+uu_x=h_0^2(u_{xt}+uu_{xx}-u_x^2)_x/3$$
is integrated as $u=f(t)-2x/t$. Then, the second equation becomes the contradictory equality $f'(t)-2f(t)/t+6x/t^2=0$. Thus, in this case there is no solution.

PIS ($L_{2.1}$). The factor-system consists of the equations

$$H'(t) + H(t)u_x = 0, \qquad u_t + uu_x = H^2(t)(u_{xt} + uu_{xx} - u_x^2)_x/3,$$

whose solution is $u = (x + u_0)/(t + a)$, $H = h_0/(t + a)$, which gives a solution of Eqs. (1.1) that is invariant with respect to the subalgebra $\langle Y_3 + aY_2\rangle$ similar to $L_{1.3}$.

PIS ($L_{2.2}$). The factor-system consists of the equations

$$h_t + (\beta t + U + hU')h_x = 0,$$

$$\beta + (g - hU'^2)h_x + (h^4U'^2h_{xx} + (2hU'U'' + 3U'^2)h^3h_x^2)_x/(3h) = 0.$$

Substitution of the solution $x - \beta t^2/2 - t(hU)' = \Phi(h)$ of the first equation into the second equation leads to the fourth-order polynomial in t:

$$\beta(\Phi' + t(hU)'')^5 + (g - HU'^2)(\Phi' + t(hU)'')^4 - h^3U'^2(\Phi'''/3 + t(hU)^{IV})(\Phi' + t(hU)'')/3$$
$$+ h^3U'^2(\Phi'' + t(hU)''')^2 - h^2U'(6hU'' + 10U')(\Phi'' + t(hU)''')(\Phi' + t(hU)'')/3$$
$$+ (2h^3U'U''' + 2h^3U''^2 + 14h^2U'U'' + 9hU'^2)(\Phi' + t(hU)'')^2/3 = 0. \tag{4.2}$$

The coefficient at t^5 vanishes when $(hU)'' = 0$. In the case $\beta = 0$, the coefficient at t^4 can also vanish for $g - HU'^2 = 0$. However, the substitution $U(h) = c_1 \pm 2\sqrt{gh}$ in (4.2) gives the contradictory equality $9g^2t^2/(8h) + P_1(t) = 0$. Therefore, we substitute the solution $U(h) = c_1/h - a$ of the equation $hU'' + 2U' = 0$ into (4.2). As a result, we obtain an equation for the function $\Phi(h)$ that does not contain t. Now, the solution $x + at - \beta t^2/2 = \Phi(h)$ of the first equation of the factor-system can be substituted into the second equation in explicit form for h: $h = H(\bar{y})$, where $\bar{y} = x + at - \beta t^2/2$. The function $H(\bar{y})$ satisfies the equation

$$\beta + (g - c_1^2/H^3)H' + c_1^2(H'''/H - 2H'H''/H^2 + H'^3/h^3)/3 = 0.$$

Integrating this equation, we find that the solution of system (1.1) for $\beta \neq 0$ is defined by the equalities

$$u = \beta t + c_1/h - a, \qquad h = H(\bar{y}), \qquad HH'' = H'^2/2 - 3/2 - 3(gH^3 + \beta(\bar{y} + c_2)H^2)/c_1^2.$$

For $\beta = 0$, the solution of Eqs. (1.1) of the simple wave type os defined by the equalities

$$u = \frac{c_1}{h} - a, \qquad \int\limits_{h_0}^{h} (3 + c_2h + c_3h^3 - 3gc_1^{-2}h^3)^{-1/2}\, dh = c_4 \pm (x + at).$$

The obtained solution is invariant with respect to the subalgebra $\langle Y_1 + \beta Y_3 - aY_2\rangle$ similar to $L_{1.2}$.

PIS ($L_{2.3}$). The factor-system consists of the equations

$$H_t + t(U + HU')H_x + (2/t)H = 0,$$

$$U - 2HU' + t^2(g - HU'^2)H_x + t^4((U' + 2HU'')H^3H_x + t^2H^4U'^2H_{xx} + t^2(2HU'U'' + 3U'^2)H^3H_x^2)_x/(3H) = 0.$$

Substitution of the solution

$$\frac{x}{t^2} + \frac{U}{2} + H\int \frac{U}{H^2}\, dH = \frac{1}{t^2}\,\Phi(\xi), \quad \xi = t^2H$$

of the first equation into the second equation leads to the equation with split variables for the functions $\Phi(\xi)$ and $U(H)$:

$$U - 2HU' + (g - HU'^2)/\Delta + (2H^3U'''/3 + 3H^2U'' + HU')/\Delta^2$$
$$+ H(2H^2U'U''' + 2H^2U''^2 + 14HU'U'' + 9U'^2 - (U' + 2HU'')(\xi\Phi'' + HV''))/(3\Delta^3)$$
$$- H(U'^2(\xi^2\Phi''' + H^2V''') + (6HU'U'' + 10U'^2)(\xi\Phi'' + HV''))/(3\Delta^4) + HU'^2(\xi\Phi'' + HV'')^2/\Delta^5 = 0,$$

$$V(H) = -\frac{U}{2} - H\int \frac{U}{H^2}\, dH, \qquad \Delta = \Phi' + V'.$$

Setting $\Phi'(\xi) = c_1$, for the function $U(H)$ we obtain Eq. (3.3), in which

$$\Delta = c_1 - \frac{U'}{2} - \frac{U}{H} - \int \frac{U}{H^2}\, dH.$$

The constant c_1 can be included in the term

$$\int \frac{U}{H^2}\, dH;$$

then $\Phi'(\xi) = 0$ and $\Phi(\xi) = -a$. After integration of Eq. (3.3), the function H is determined from the relation

$$\frac{x+a}{t^2} = -\frac{U}{2} - H \int \frac{U}{H^2}\, dH,$$

which gives a solution of Eqs. (1.1) that is invariant with respect to the subalgebra $\langle Y_4 + 2aY_2\rangle$ similar $L_{1.4}$.

PIS ($L_{2.4}$). The factor-system consists of the equations

$$H_t + (\beta t + y^{1/2}(U + HU'))H_x + (3/2)y^{-1/2}HU = 0,$$

$$\beta + U^2/2 - 3HUU'/2 + gH + y(g - HU'^2)H_x + (y^4(2HU'U'' + 3U'^2)H^3H_x^2$$

$$+ y^4H^4U'^2H_{xx} + y^3(3HUU'' + 5HU'^2 + 5UU')H^3H_x/2 + y^2H^3U^2/2)_x/(3yH) = 0.$$

From the solution of the first equation

$$ty^{-1/2} + \frac{2}{3}\,(HU)^{1/3} \int (HU)^{-4/3}\, dH = y^{-1/2}\Phi(\xi), \quad \xi = y^{1/2}(HU)^{1/3},$$

we find the quantities

$$H_x = -\frac{3HUV}{2y\Delta}, \qquad H_{xx} = \frac{3}{y^2}\, HU\Big(\frac{5V}{4\Delta} + \frac{V}{\Delta^2} - \frac{3HU}{4\Delta^3}\,(HU)''V^3 - \frac{(HU)^{1/3}}{\Delta^3}\,\xi\Phi''\Big),$$

$$V = (HU)^{1/3}\Big(\Phi' - \frac{2}{3}\int (HU)^{-4/3}\, dH\Big), \qquad \Delta = (HU)'V - 2.$$

Substitution of the above quantities into the second equation of the factor-system yields an equation for the functions $\Phi(\xi)$ and $U(H)$, which is not given here to save space. Setting $\Phi'(\xi) = c_1$ in this equation, for the function $U(H)$ we obtain Eq. (3.4), in which

$$v = (HU)^{1/3}\Big(c_1 - \frac{2}{3}\int (HU)^{-4/3}\, dH\Big).$$

The constant c_1 can be included in the term

$$\int (HU)^{-4/3}\, dH;$$

then, $\Phi'(\xi) = 0$ and $\Phi(\xi) = -a$. After integration of Eq (3.4), the functions H are determined from the relation

$$(t+a)y^{-1/2} = -\frac{2}{3}\,(HU)^{1/3}\int (HU)^{-4/3}\, dH,$$

which gives a solution of Eqs. (1.1) that is invariant with respect to the subalgebra $\langle Y_4 + a(Y_1 + \beta Y_3)\rangle$, similar $L_{1.4}$.

PIS ($L_{2.5}$). The factor-system consists of the equations

$$H_t + (x/t + t(U + HU'))H_x + 3H/t = 0,$$

$$2U - 3HU' + t^2(g - HU'^2)H_x + t^2(t^4H^4U'^2H_{xx}$$

$$+ t^4(2HU'U'' + 3U'^2)H^3H_x^2 + t^2(5U' + 3HU'')H^3H_x + 2H^3)_x/(3H) = 0.$$

Substitution of the solution

$$\frac{x}{t^2} + \frac{U}{3} + \frac{4}{9}\,H^{1/3}\int H^{-4/3}U\, dH = \frac{1}{t}\,\Phi(\xi), \quad \xi = tH^{1/3}$$

of the first equation into the second equation leads to the following equation for the functions $\Phi(\xi)$ and $U(H)$:

$$2U - 3HU' + (g + 2H - HU'^2)/\Delta + (H^3U''' + (17/3)H^2U'' + 5HU')/\Delta^2$$

$$+H^3U'^2(V'' + H^{-5/3}(\xi\Phi'' - 2\Phi')/9)^2/\Delta^5 + H(2H^2U'U''' + 2H^2U''^2 + 14HU'U''$$

$$+ 9U'^2 - (3H^2U'' + 5HU')(V'' + H^{-5/3}(\xi\Phi'' - 2\Phi')/9))/(3\Delta^3)$$

$$- H^2(HU'^2(V''' + H^{-8/3}(\xi^2\Phi''' - 6\xi\Phi'' + 10\Phi')/27)/(3\Delta^4)$$

$$+ (6HU'U'' + 10U'^2)(V'' + H^{-5/3}(\xi\Phi'' - 2\Phi')/9)) = 0,$$

$$V(H) = -\frac{U}{3} - \frac{4}{9}\,H^{1/3}\int H^{-4/3}U\,dH, \qquad \Delta = V' + \frac{1}{3}\,H^{-2/3}\Phi'.$$

Setting $\Phi'(\xi) = c_1$, for the function $U(H)$ we obtain Eq. (3.5), in which

$$\Delta = \frac{1}{3}\,H^{-2/3}\Big(c_1 - \int H^{-4/3}U\,dH\Big) - \frac{U'}{3} - \frac{4}{9}\,\frac{U}{H}.$$

The constant c_1 can be included in the term $\int H^{-4/3}U\,dH$; then $\Phi'(\xi) = 0$ and $\Phi(\xi) = -a$. After integration of Eq. (3.5), the function H is determined from the relation

$$\frac{x + at}{t^2} = -\frac{U}{3} - \frac{4}{9}\,H^{1/3}\int H^{-4/3}U\,dH,$$

which gives a solution of Eqs. (1.1) that is invariant with respect to the subalgebra $\langle Y_4 + aY_3\rangle$ similar to $L_{1.4}$.

We thank N. I. Makarenko, who has drawn our attention to the given problem.

This work was supported by the Russian Foundation for Basic Research (Grant Nos. 02-01-00550 and 05-01-97910) and the foundation Leading Scientific Schools of Russia (Grant No. NSh. 440.2003.1).

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